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Static and Dynamic, Local and Global, Bifurcations in Nonlinear Autonomous Structural Systems

A. N. Kounadis*
National Technical University of Athens, Athens 10682, Greece

Discrete damped or undamped gradient systems described by nonlinear autonomous ordinary differential equations (ODE) of motion are examined in detail. Emphasis is given to the relationship between static with possibly existing dynamic bifurcations. Criteria for dynamic bifurcations and stability of precritical, critical, and postcritical states associated with the nature of Jacobian eigenvalues are also presented. Cases of discrepancies between local and global dynamic analysis regarding the stability of equilibriums are reported for the first time. Finally, using a simple energy criterion, exact dynamic buckling loads for vanishing but nonzero damping are readily obtained.

Nomenclature

a_i	= coefficients of the characteristic equation of the Jacobian matrix, $i = 1, \ldots, n$
$[b_{ij}]$	= inverse of the positive definite matrix of the kinetic energy, $i, j = 1, \ldots, n$
c_i	= coefficients of linear viscous damping, $i = 1, 2$
$[c_{ij}]$	= Rayleigh, non-negative definite, dissipation matrix, $i, j = 1, \ldots, n$
E	= total energy
F	= Rayleigh dissipation function
K	= kinetic energy
m_i	= concentrated masses, $i = 1, 2$
Q	= vector with components Q_i, being quadratic forms in $\dot{q}$, $Q(Q_1, \ldots, Q_n)^T$
Q^*	= vector with components Q_i^*, being quadratic forms in $\dot{q}$, $Q^*(Q_1^*, \ldots, Q_n^*)^T$
q	= vector with components the generalized displacements q_i, $q(q_1, \ldots, q_n)^T$
q_i^0, q_i^D	= generalized displacements at $t = 0$ and at the critical state, respectively
S_i, S_i^*	= nonsymmetric and symmetric matrix, respectively, $i, j = 1, \ldots, n$
V, V_i, V_{ij}	= total potential energy, its first and second partial derivative, $i = 1, \ldots, n$
$[V_{ij}]$	= symmetric matrix of the second variation, $i, j = 1, \ldots, n$
Y	= nonlinear vector function with components Y_i, $Y(Y_1, \ldots, Y_{2n})^T$
Y_y	= Jacobian square matrix of order $2n$, $\partial Y/\partial y$
y	= state vector with components y_i, $y(y_1, \ldots, y_{2n})^T$
y^E, y^R	= state vector at an equilibrium and a nonequilibrium point, respectively
$\tilde{y}$	= vector with components $y_{n+1}, y_{n+2}, \ldots, y_{2n}$
$[\alpha_{ij}]$	= $[b_{ij}]^{-1}$, $i, j = 1, \ldots, n$
$\lambda, \tilde{\lambda}_D$	= step load of infinite duration and dynamic buckling load for vanishing (nonzero) damping, respectively
λ_D, λ_{DD}	= dynamic buckling load of undamped and damped system, respectively
λ_s, λ_c	= static limit point and bifurcational load, respectively

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*Professor of Civil Engineering.

I. Introduction

IN this work attention is given to the application of certain basic ideas and concepts of the theory of dynamical systems to nonlinear dynamic buckling and stability problems of autonomous structural systems in a comprehensive but mathematically rigorous way. In this analysis, extending previous studies by Kounadis and Raftoyiannis[1] and Kounadis,[2] the following definitions and clarifications have to be made.

We briefly repeat here that nonlinear dynamic buckling, being associated with the criterion of Lagrange (boundedness of solution), is defined as that state for which a small change in the load produces an "unbounded" motion or more precisely a large change in the response. Then an escaped motion is initiated either through a saddle point in the case of damped systems or through a nonequilibrium point lying in the vicinity of a saddle in case of undamped systems. The minimum possible load corresponding to the aforementioned state is defined as the dynamic buckling load.[3,4]

One should also clarify what is meant by dynamic bifurcation. This type of bifurcation is defined as a sudden qualitative change of the system response occurring at a certain value of a smoothly varying control parameter. Emphasis is given to both local and global bifurcations. The first bifurcations can be studied by a local (linear) analysis with the aid of the Jacobian matrix eigenvalues, whereas the second ones can be explored only by a global (nonlinear) dynamic analysis.

The main objectives of this investigation dealing with autonomous damped or undamped systems are the following: 1) to discuss the relation of the different types of simple or distinct static bifurcations (asymmetric, stable and unstable symmetric) emanating from a trivial prebuckling path with corresponding, possibly existing, dynamic bifurcations occurring at the same critical load; 2) to find out, in connection with the previous types of static bifurcations, if there are cases in which dynamic buckling or dynamic bifurcation occurs before or after static buckling; and 3) to explore if there exist equilibrium points that are unstable when a global (nonlinear) analysis is employed but stable according to the local analysis having Jacobian eigenvalues with negative real parts.

II. General Formulation

Consider a general n-degree-of-freedom discrete structural system under a step loading of infinite duration λ. The nonlinear autonomous ordinary differential equations (ODE) of Lagrange governing its motion in terms of generalized displacements q_i and generalized velocities $\dot{q}_i (i = 1, \ldots, n)$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = 0, \qquad (i = 1, \ldots, n) \qquad (1)$$

where the dots denote differentiation with respect to time t; $K = \frac{1}{2}\,\dot{q}^T[\alpha_{ij}]\dot{q}$ is the positive definite function of the total kinetic energy with nondiagonal elements $\alpha_{ij} = \alpha_{ij}(q_1, \ldots, q_n)$ and $\dot{q}^T$, being the transpose of the vector $\dot{q}$ with components $\dot{q}_i$ $(i = 1, 2, \ldots, n)$; $V = V(q_1, \ldots, q_n; \lambda)$ is the total potential energy, being a linear function of the step loading λ; and $F = \frac{1}{2}\,\dot{q}^T[c_{ij}]\dot{q}$ is the non-negative definite (viscous) dissipation function of Rayleigh with coefficients c_{ij} that might be functions of q_i[i.e., $c_{ij} = c_{ij}(q_1, \ldots, q_n)$]. The loading λ is considered as the control parameter for the occurrence of static and dynamic bifurcation, as it was defined in Sec. I. From the point of view of topology, dynamic bifurcations correspond to those values of the control parameter λ for which the response of the dynamical system becomes structurally unstable.[5]

The previous nonlinear system under the same loading applied statically is assumed to exhibit the three types of simple bifurcations (i.e., stable and unstable symmetric and asymmetric) emanating from a trivial prebuckling path that corresponds to curves a, b, and c of Fig. 1. This can always be realized by modifying properly the structural form of the strain energy incorporated into V.

If the previous perfect system is initially $(t_0 = 0)$ at rest [i.e., $q_i(t_0 = 0) = \dot{q}_i(t_0 = 0) = 0,\ i = 1, \ldots, n$], we can write the total energy E, valid at any time $t > 0$, as follows:

$$E = K + V + 2\int_0^t F\,\mathrm{d}t = 0 \tag{2}$$

This equation can be used as a measure for checking the accuracy of numerical solutions, particularly if they are obtained after large time.

From the expression of the total kinetic energy $K = \frac{1}{2}\dot{q}^T[\alpha_{ij}]\dot{q}$ we get

$$\frac{\partial K}{\partial \dot{q}} = [\alpha_{ij}]\dot{q}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}}\right) = [\alpha_{ij}]\ddot{q} + [\dot{\alpha}_{ij}]\dot{q} \tag{3}$$

where $[\dot{\alpha}_{ij}]\dot{q} = Q$ is a column matrix with ith element given by

$$Q_i = [\dot{q}_1, \ldots, \dot{q}_n] \times \begin{bmatrix} \frac{\partial \alpha_{i1}}{\partial q_1} & \frac{\partial \alpha_{i2}}{\partial q_1} & \cdots 0 \cdots & \frac{\partial \alpha_{in}}{\partial q_1} \\ \frac{\partial \alpha_{i1}}{\partial q_2} & \frac{\partial \alpha_{i2}}{\partial q_2} & \cdots 0 \cdots & \frac{\partial \alpha_{in}}{\partial q_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \alpha_{i1}}{\partial q_n} & \frac{\partial \alpha_{i2}}{\partial q_n} & \cdots 0 \cdots & \frac{\partial \alpha_{in}}{\partial q_n} \end{bmatrix} \times \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} \tag{4}$$

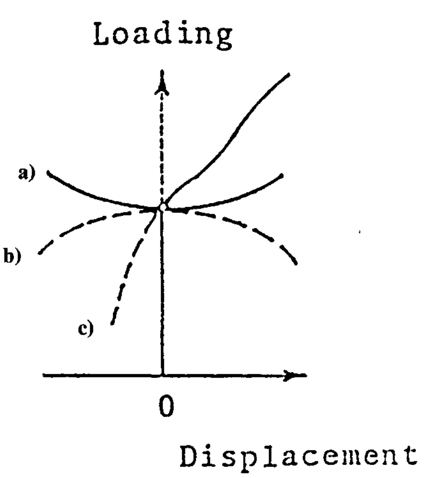

Fig. 1 Types of bifurcation points: a) stable symmetric; b) unstable symmetric; and c) asymmetric.

where $\partial \alpha_{ii}/\partial q_k = 0$ $(i,\ k = 1, 2, \ldots, n)$, since the diagonal terms of the matrix $[\alpha_{ij}]$, being only functions of masses, are independent of $q_i (i = 1, \ldots, n)$. One can also obtain

$$\frac{\partial K}{\partial q_i} = \frac{1}{2}\,\dot{q}^T\left[\frac{\partial \alpha_{ij}}{\partial q_i}\right]\dot{q} \tag{5}$$

where due to $\partial \alpha_{ii}/\partial q_k = 0$ the matrix $[\partial \alpha_{ij}/\partial q_i]$ is given by

$$\begin{bmatrix} 0 & \frac{\partial \alpha_{12}}{\partial q_i} & \frac{\partial \alpha_{13}}{\partial q_i} & \cdots & & \cdots & \frac{\partial \alpha_{1n}}{\partial q_i} \\ & 0 & \frac{\partial \alpha_{23}}{\partial q_i} & \frac{\partial \alpha_{24}}{\partial q_i} & \cdots & \cdots & \frac{\partial \alpha_{2n}}{\partial q_i} \\ & & 0 & \frac{\partial \alpha_{34}}{\partial q_i} & \frac{\partial \alpha_{35}}{\partial q_i} \quad \cdots & \cdots & \frac{\partial \alpha_{3n}}{\partial q_i} \\ \text{Symmetric} & & & & \vdots & & \vdots \\ & & & & \frac{\partial \alpha_{n-2,n-1}}{\partial q_i} & & \frac{\partial \alpha_{n-2,n}}{\partial q_i} \\ & & & & 0 & & \frac{\partial \alpha_{n-1,n}}{\partial q_i} \\ & & & & & & 0 \end{bmatrix} \tag{6}$$

Using relations (4–6), one can determine the column matrix Q^* with ith element given by

$$Q_i^* = Q_i - \frac{\partial K}{\partial q_i} = \dot{q}^T S_i \dot{q} \tag{7}$$

where S_i is a square nonsymmetric matrix equal to the difference of the square matrices associated with Q_i [Eq. (4)] and $\partial K/\partial q_i$[Eq. (5)]. Writing S_i as the sum of a symmetric and a skew symmetric matrix, we can always transform $\dot{q}^T S_i \dot{q}$ into a quadratic form; that is,

$$\dot{q}^T S_i \dot{q} = \dot{q}^T S_i^* \dot{q}, \qquad (S_i^* = S_i^{*T}) \tag{8}$$

Using Eqs. (3–8), we can write Eqs. (1) in matrix-vector form as follows:

$$[\alpha_{ij}]\ddot{q} + Q^* + [c_{ij}]\dot{q} + \frac{\partial V}{\partial q} = 0 \tag{9}$$

where $\partial V/\partial q$ is a column matrix with ith element $\partial V/\partial q_i$ $(i = 1, \ldots, n)$. Since $[\alpha_{ij}]$ is always a positive definite matrix, one can obtain

$$\ddot{q} = -[b_{ij}]Q^* - [\tilde{c}_{ij}]\dot{q} - [b_{ij}]\frac{\partial V}{\partial q}$$

where (10)

$$[b_{ij}] = [\alpha_{ij}]^{-1} \qquad \text{and} \qquad [\tilde{c}_{ij}] = [b_{ij}][c_{ij}]$$

The set of n second-order Lagrange equations, Eqs. (10) can be replaced by a set of $2n$ first-order Hamiltonian equations. Setting

$$y_1 = q_1,\ y_2 = q_2, \ldots, y_n = q_n$$

and (11)

$$y_{n+1} = \dot{q}_1,\ y_{n+2} = \dot{q}_2, \ldots, y_{2n} = \dot{q}_n$$

Equations (10) can be written as follows[6]:

$$\dot{y}_i = Y_i(y_1, \ldots, y_{2n}; \lambda), \qquad i = 1, 2, \ldots, 2n \tag{12}$$

where

$$
\begin{aligned}
& Y_1 = y_{n+1},\ Y_2 = y_{n+2},\ \ldots,\ Y_n = y_{2n} \\
& y_{n+1} = -\sum_{i=1}^{n} b_{1i}\tilde{y}^T S_i^* \tilde{y} - \sum_{i=1}^{n} \tilde{c}_{1i} y_{n+i} - \sum_{i=1}^{n} b_{1i}\frac{\partial V}{\partial y_i} \\
& y_{n+2} = -\sum_{i=1}^{n} b_{2i}\tilde{y}^T S_i^* \tilde{y} - \sum_{i=1}^{n} \tilde{c}_{2i} y_{n+i} - \sum_{i=1}^{n} b_{2i}\frac{\partial V}{\partial y_i} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& y_{2n} = -\sum_{i=1}^{n} b_{ni}\tilde{y}^T S_i^* \tilde{y} - \sum_{i=1}^{n} \tilde{c}_{ni} y_{n+i} - \sum_{i=1}^{n} b_{ni}\frac{\partial V}{\partial y_i}
\end{aligned}
\tag{13}
$$

with $\tilde{y}$ being a vector with components $y_{n+1}, y_{n+2}, \ldots, y_{2n}$.

Equations (12) can be written in matrix-vector form as follows:

$$\dot{y} = Y(y;\lambda) \tag{14}$$

where $y = (Y_1, \ldots, y_{2n})^T$ is the state vector, being continuously dependent on t and λ; the nonlinear vector function $Y = (Y_1, \ldots, Y_{2n})^T$ is assumed to satisfy the Lipschitz conditions,[7] at least in the domain of interest, for all t and λ. This implies that $y_i(t)$ $(i = 1, \ldots, n)$ belongs to the class of functions C^2.

Setting the left-hand side of Eq. (14) equal to zero, one can obtain the equilibrium states y^E as follows:

$$Y(y^E;\lambda) = 0 \tag{15}$$

whose equivalent form of Eqs. (13) yields

$$V_i(y^E;\lambda) = 0 \qquad (i = 1, \ldots, n) \tag{16}$$

III. Local Bifurcations

Using a local analysis one can establish local static and dynamic bifurcations as well as discuss stability of equilibriums.

Static Bifurcations

Since the static bifurcations are associated with a trivial fundamental equilibrium path, Eq. (15), as well as Eq. (16), is satisfied by the zero solution $y = 0$. This is also true for Eq. (14), regardless of the value of λ; that is,

$$Y(0;\lambda) = 0 \qquad \text{or} \qquad V_i(0;\lambda) = 0 \tag{17}$$

But the system also experiences another solution at the critical state, $\lambda = \lambda_c$, which is different from zero, $y \neq 0$; that is, it displays a static bifurcation associated with a postbuckling equilibrium path. The prebuckling equilibrium path is stable up to the critical value λ_c (i.e., for $\lambda < \lambda_c$), regardless of the type of static bifurcation. This is assured when the second variation of the total potential energy, $\delta^2 V$, is positive definite or equivalently when the symmetric matrix $[V_{ij}]$ associated with it is positive definite. At $\lambda = \lambda_c$ the matrix $[V_{ij}]$ becomes positive semidefinite, because one of its eigenvalues vanishes. Then the static critical loads are obtained by the buckling equation

$$|V_{ij}(0;\lambda)| = 0 \tag{18}$$

Recall that the elements $V_{ij} = \partial^2 V/\partial q_i \partial q_j (i, j = 1, 2, \ldots, n)$ of the matrix $[V_{ij}]$ are linear functions of λ, and therefore the determinant $|V_{ij}(0;\lambda)|$ is an algebraic polynomial of nth degree in λ.

Dynamic Bifurcations

Dynamic bifurcations, in some cases, cannot be established by a local analysis. Then a global analysis must be employed (global bifurcations). For instance, dynamic bifurcations with trajectories passing through saddle points or cases where closed orbits become nonhyperbolic (at least one characteristic multiplier has unit modulus) can be detected only by using a global analysis.[8] Such an analysis is also the only safe way for exploring chaotic phenomena or chaoslike phenomena.[9-11]

For the gradient damped system under discussion, the case of a limit cycle response is excluded. Hence, its response is associated[11] either with point attractors (sinks) or with point repellers (sources) or saddles.

A local dynamic analysis is associated with the study of the nature of the eigenvalues of the Jacobian matrix evaluated at a singular (equilibrium) point y^E or nonsingular (nonequilibrium) point y^R, whose stability we wish to discuss. As stated earlier, y^R is a dynamic nonequilibrium critical point through which dynamic buckling occurs in case of multi-degree-of-freedom undamped systems.[2]

For the stability of a point $y^0(= y^E$ or $y^R)$ we can examine the motion in its neighborhood. A Taylor's expansion of $Y(y;\lambda)$ around y^0 yields

$$\dot{y} = Y_y(y^0;\lambda)y + \text{higher order terms} \tag{19}$$

where $Y_y(y^0;\lambda) = \partial Y(y^0;\lambda)/\partial y$ is the Jacobian matrix evaluated at y^0.

According to the Haztman-Gzobman theorem,[11] the behavior of the nonlinear system (14) *near* a hyperbolic equilibrium point y^E (i.e., a point in which none of the Jacobian eigenvalues has zero real part) is qualitatively determined by the behavior of the linear system $\dot{y} = Y_y(y^E;\lambda)\ y$, where y^E is assumed to be located at the origin; otherwise it can be translated to the origin by setting $y \to y - y^E$.

Using relations (11–13), one can show that the previous Jacobian matrix is a partitioning matrix composed from four square submatrices of order n; that is,

$$
Y_y(y^0;\lambda) = \left[\begin{array}{ccc|ccc}
 & 0 & & & I_n & \\
\hline
\dfrac{\partial Y_{n+1}}{\partial y_1} & \cdots & \dfrac{\partial Y_{n+1}}{\partial y_n} & \dfrac{\partial Y_{n+1}}{\partial y_{n+1}} & \cdots & \dfrac{\partial Y_{n+1}}{\partial y_{2n}} \\
\vdots & & \vdots & \vdots & & \vdots \\
\dfrac{\partial Y_{2n}}{\partial y_1} & \cdots & \dfrac{\partial Y_{2n}}{\partial y_n} & \dfrac{\partial Y_{2n}}{\partial y_{n+1}} & \cdots & \dfrac{\partial Y_{2n}}{\partial y_{2n}}
\end{array}\right]
= \begin{bmatrix} 0 & I_n \\ -[\tilde{V}_{ij}] & -[\tilde{c}_{ij}] \end{bmatrix} \tag{20}
$$

where by 0 and I_n we denote the zero and the identity matrix of order $n \times n$, whereas due to Eqs. (13)

$$[\tilde{V}_{ij}] = [b_{ij}][V_{ij}] \tag{21}$$

The characteristic equation of the Jacobian matrix (20) evaluated at $y^0 = y^E$ is given by

$$|y_y(y^E;\lambda) - \rho I_{2n}| = |\rho^2 I_n + \rho[\tilde{c}_{ij}] + [\tilde{V}_{ij}]| = 0 \tag{22}$$

The last equation is also equivalent to

$$|\rho^2[\alpha_{ij}] + \rho[c_{ij}] + [V_{ij}]| = 0 \tag{23}$$

Expansion of the determinant (22) leads to the following characteristic equation:

$$f(\rho) = \rho^{2n} + a_1\rho^{2n-1} + a_2\rho^{2n-2} + \cdots + a_{2n-1}\rho + a_{2n} = 0 \tag{24}$$

where

$$
\begin{aligned}
a_1 &= -\operatorname{tr} Y_y(y^E;\lambda) = \sum_{i=1}^{n} \tilde{c}_{ii} \\
a_{2n} &= \det Y_y = \det[\tilde{V}_{ij}] = \frac{\det[V_{ij}]}{\det|\alpha_{ij}|}
\end{aligned}
\tag{25}
$$

Then Eq. (18) is equivalent to

$$\det Y_y(0;\lambda) = 0 \quad \text{or} \quad |Y_y(0;\lambda)| = 0 \tag{26}$$

Nature of Eigenvalues

Dynamic bifurcations and stability of equilibriums can be established by studying the nature of the Jacobian matrix eigenvalues, being functions of the control parameter λ. It is widely accepted that the results of the local analysis are also valid for the original nonlinear equations of motion (14), provided that the case $\lambda = \lambda_c$ is excluded. However, as will be shown later, this is not always true.

As λ increases from zero to its critical value λ_c (i.e., for $0<\lambda<\lambda_c$) the term a_{2n} of Eq. (24) remains positive since both matrices $[V_{ij}]$ and $[\alpha_{ij}]$ are positive definite. At $\lambda = \lambda_c$ the matrix $[V_{ij}]$ becomes positive semidefinite, and then Eq. (24) yields $\rho = 0$; i.e., one Jacobian eigenvalue becomes zero. As is known, $\rho = 0$ implies a dynamic bifurcation; hence the static bifurcation at $\lambda = \lambda_c$ is also a dynamic (local) bifurcation provided that dynamic buckling, associated with a global bifurcation, does not take place before static buckling (i.e., for $\lambda<\lambda_c$).

Regarding the stability of the precritical trivial states, one can observe the following.

Let us first consider the case of zero damping; that is, $c_{ij} = 0$ for $i, j = 1, \ldots, n$. Then Eq. (23) becomes

$$|\rho^2[\alpha_{ij}] + [V_{ij}]| = 0 \tag{27}$$

Since $[\alpha_{ij}]$ is a positive definite matrix, there is always a positive definite matrix $[\alpha_{ij}]^{1/2}$ for which $\{[\alpha_{ij}]^{1/2}\}^2 = [\alpha_{ij}]$. Then one can write

$$\rho^2[\alpha_{ij}] + [V_{ij}] = [\alpha_{ij}]^{1/2}\{\rho^2 I_n + [\alpha_{ij}]^{-1/2}[V_{ij}][\alpha_{ij}]^{-1/2}\}[\alpha_{ij}]^{1/2} \tag{28}$$

It is clear that the eigenvalues of matrix $[\alpha_{ij}]^{-1}[V_{ij}] = [\tilde{V}_{ij}]$, resulting from Eq. (27), are also eigenvalues of the real symmetric matrix $[\alpha_{ij}]^{-1/2}\cdot[V_{ij}]\cdot[\alpha_{ij}]^{-1/2}$, and vice versa. Since the latter matrix is symmetric, the eigenvalues are real for $\lambda<\lambda_c$ because $[V_{ij}]$ is a positive definite matrix.

If ρ^2 is an eigenvalue of the Jacobian matrix with corresponding (real) eigenvector r, then by virtue of relation (27), one can write

$$\{\rho^2[\alpha_{ij}] + [V_{ij}]\}r = 0 \tag{29}$$

This equation after premultiplication by r^T yields

$$\rho^2 = -\frac{r^T[V_{ij}]r}{r^T[\alpha_{ij}]r} \tag{30}$$

Since both matrices $[V_{ij}]$ and $[\alpha_{ij}]$ are positive definite, the corresponding quadratic forms in Eq. (30) are positive definite. Hence all eigenvalues of the Jacobian for $\lambda<\lambda_c$ are pairs of purely imaginary roots of Eq. (24) from which all terms with odd powers are missing.

Let $\tilde{\rho}_i (i = 1, \ldots, n)$ be the eigenvalues of the matrix $[\tilde{V}_{ij}]$ resulting from equation $|\tilde{\rho}I_n - [\tilde{V}_{ij}]| = 0$. Then the Jacobian eigenvalues ρ_j $(j = 1, \ldots, 2n)$ are related to the eigenvalues of the matrix $[\tilde{V}_{ij}]$ through the formula

$$\rho_j = \pm i\sqrt{\tilde{\rho}_i} \qquad (\tilde{\rho}_i>0 \text{ for all } i = 1, \ldots, n) \tag{31}$$

where $j = 1, 2, \ldots, 2n$ and $i = \sqrt{-1}$.

The local solution, associated with pairs of purely imaginary eigenvalues, shows that the undamped system undergoes nonlinear bounded oscillations with closed trajectories around the origin that is a center. This means that all precritical trivial states $y^E = 0$ are stable. Whether this result is valid when a nonlinear dynamic analysis is employed will be discussed later.

When damping is included instead of Eq. (29), we have

$$\rho^2[\alpha_{ij}]r + \rho[c_{ij}]r + [V_{ij}]r = 0 \tag{32}$$

Premultiplying Eq. (32) by r^T yields a second degree algebraic equation in ρ from which we get

$$\rho = \frac{-r^T[c_{ij}]r \pm \sqrt{\{r^T[c_{ij}]r\}^2 - 4\{r^T[\alpha_{ij}]r\}\{r^T[V_{ij}]r\}}}{2\{r^T[\alpha_{ij}]r\}} \tag{33}$$

Since the quadratic forms $r^T[V_{ij}]r$ (for $\lambda<\lambda_c$) and $r^T[\alpha_{ij}]r$ are positive definite, whereas $r^T[c_{ij}]r$ is non-negative definite, both roots of Eq. (33) for $|c_{ij}| \neq 0$ are, in general, negative. If ρ is not real, r^T should be replaced by its complex conjugate $\bar{r}^T$. In case of structural (i.e., small) damping, the quantity under the radical is negative, and therefore Eq. (33) has a pair of complex conjugate roots with negative real part.

When all Jacobian eigenvalues have negative real parts, all trivial precritical states, $y^E = 0$, are asymptotically stable; then the motion of the damped system converges toward the trivial state.

IV. Global Analysis

The important question that now arises is whether the earlier results of local analysis regarding dynamic bifurcations and stability of precritical, critical, and postcritical trivial states $y^E = 0$, are also valid when a nonlinear (global) dynamic analysis is employed. Namely, are there cases where dynamic buckling occurs before or after static buckling? In the sequel an attempt is made to explain theoretically these phenomena by extending previous ideas valid for limit point systems[2,9] to the case of unstable bifurcational perfect systems due to either an unstable symmetric or an asymmetric branching point lying on a trivial fundamental path. However, before answering this question a theoretical explanation of the mechanism of dynamic buckling will be given next.

Mechanism of Dynamic Buckling

The perfect bifurcational system, after introducing a small imperfection, becomes a limit point system with corresponding load $\lambda_s<\lambda_c$, where λ_c is the static bifurcational load of the perfect system.

The previous limit point system under step loading of infinite duration exhibits dynamic buckling at a load λ_D or λ_{DD} (when damping is included) having as upper bound the load λ_s and as a lower bound the load $\tilde{\lambda}_D$ for which the total potential energy V vanishes at a saddle point of the unstable postbuckling path; that is,[2]

$$\tilde{\lambda}_D<\lambda_D<\lambda_{DD}<\lambda_s<\lambda_c \tag{34}$$

For $\lambda<\lambda_{DD}$ the system exhibits a point attractor response. The precritical trivial equilibrium states, which are asymptotically stable, capture the motion that for $t\to\infty$ converges toward them. Each stable equilibrium point has its own basin of attraction; any motion originating in it leads after the decay of transients as $t\to\infty$ to the corresponding stable equilibrium point. On the contrary, the equilibrium points of the postbuckling path are saddle points that in one or more directions act as repellors, whereas in the remaining directions as point attractors. The second variation $\delta^2 V$ evaluated at a saddle point is indefinite, which, due to Eqs. (24) and (25), leads to the conclusion that at least one pair of the Jacobian eigenvalues has positive real part. Each saddle point is associated with invariant smooth inset (stable) and outset (unstable) manifolds asymptotic to it. The inset manifold acts as attractor capturing the motion, whereas the outset manifold acts as repellor sending away the motion (dynamic buckling). The inset manifold is identified as the set of trajectories asymptotic to a saddle as $t\to\infty$, whereas the outset is the set of trajectories asymptotic to a saddle as $t\to-\infty$. The inset and outset smooth manifolds are approximated by straight lines in two dimensions, whereas in higher dimensions they may be curves, smooth surfaces, or hypersurfaces.

Consider now at each level of the loading λ the basin of attraction of the corresponding stable equilibrium point as well as the invariant inset and outset manifolds corresponding to the same loading saddle[2,9] (see Fig. 2). As λ increases from zero, there is an overall tendency of the boundary of the basin of attraction of the corresponding stable equilibrium point y^E to approach the unstable postbuckling path. At a certain value of λ there corresponds a stable equilibrium point with a basin of attraction whose boundary touches the inset and outset manifolds corresponding to the same loading saddle. Then the motion is captured by the inset (stable) manifold, and when it reaches the saddle, it escapes through its outset (unstable) manifold. Thus, the saddle is the threshold through which dynamic buckling takes place. The corresponding to that state loading is defined as the dynamic buckling load λ_{DD} that according to inequality (34) has as lower bound the load $\tilde{\lambda}_D$ for which the total potential energy V vanishes at a saddle of the postbuckling path. Indeed, given that dynamic buckling takes place through a saddle point, the kinetic energy K at that instant becomes zero. Then, Eq. (2) yields

$$V(q;\lambda) = -2\int_0^t F \, \mathrm{d}t = -\int_0^t \dot{q}^T [c_{ij}] \dot{q} \, \mathrm{d}t' < 0 \tag{35}$$

In case of structural (small) damping, the condition

$$V(q, \lambda) = 0, \qquad q = (q_1, \ldots, q_n)^T \tag{36}$$

along with the equilibrium equations

$$\frac{\partial V(q, \lambda)}{\partial q} = 0 \tag{37}$$

yield [due to Eq. (35)] a lower bound dynamic buckling estimate $\tilde{\lambda}_D$ provided that the solution $(\tilde{q}_D, \tilde{\lambda}_D)$ renders $\delta^2 V$ indefinite. Note that $\tilde{\lambda}_D$ approaches the accurate dynamic buckling load λ_{DD} as damping decreases, excluding the case of zero damping. For vanishing but nonzero damping, $[c_{ij}] \to 0$, conditions (36) and (37) yield

$$\tilde{\lambda}_D \to \lambda_{DD} \tag{38}$$

This is a new finding very important for structural design purposes. For zero damping of systems with more than one degree of freedom, Eqs. (36) and (37) yield $\tilde{\lambda}_D$, which is a lower bound[2] of λ_D.

A typical explanatory sketch of such a mechanism for an imperfect two-degree-of-freedom damped system is shown in Fig. 2. Note that the inset and outset manifolds of the saddle D through which dynamic buckling occurs are approximated by straight lines tangent to the boundary of the basin of attraction of the equilibrium point with $\lambda = \lambda_{DD}$.

Subsequently, we will extend the previous dynamic buckling mechanism to the case of bifurcational perfect systems associated either with symmetric or asymmetric branching points.

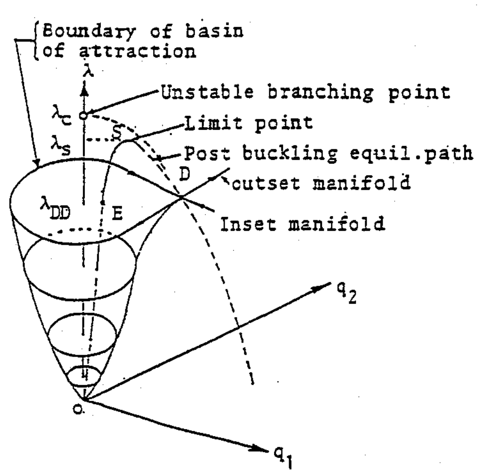

Fig. 2 Boundaries of basins of attraction of stable equilibrium points and the inset and outset manifolds of a saddle D.

When the imperfection parameter tends to zero (perfect system), then $\lambda_s \to \lambda_c$ and the total potential energy evaluated at the postbuckling path tends also to zero; that is, $V(0; \lambda) \to 0$, and therefore it is reasonable to expect $\tilde{\lambda}_D \to \lambda_c$, which should lead to

$$\tilde{\lambda}_D \to \lambda_D \to \lambda_{DD} \to \lambda_s \to \lambda_c \tag{39}$$

It will also be sought whether dynamic buckling may occur before or after static buckling.

The aforementioned questions will be discussed by analyzing simple autonomous systems of one and two degrees of freedom.

One-Degree-of-Freedom Systems

If such an undamped system exhibits an unstable branching point lying on a trivial fundamental path, Eq. (9) due to $Q_1 = 0$ becomes

$$\alpha_{11}\ddot{q}_1 + \frac{\partial V}{\partial q_1} = 0 \tag{40}$$

where $V = V(q_1, \lambda)$ is a nonlinear function of q_1 and linear function of λ.

If a small initial geometric imperfection q_1^0 is introduced, the system under static load loses its stability via a limit point with $\lambda_s < \lambda_c$. If the last system is subjected to a step loading of infinite duration, an "unbounded" motion leading to dynamic buckling occurs when the total potential energy $V(q_1, \lambda; q_1^0)$ vanishes at a saddle of the unstable postbuckling path.[2-4] The dynamic buckling load $\tilde{\lambda}_D$ and the respective dynamic displacement q_1^D are obtained by solving the system of equations

$$V(q_1, \lambda; q_1^0) = \frac{\mathrm{d}V}{\mathrm{d}q_1}(q_1, \lambda; q_1^0) = 0$$
$$\left[\text{with } \frac{\mathrm{d}^2 V}{\mathrm{d}q_1^2}(q_1^D, \tilde{\lambda}_D; q_1^0) < 0\right] \tag{41}$$

From the first of Eqs. (41) one can establish

$$\lambda = \lambda(q_1; q_1^0) \tag{42}$$

which upon introducing it to $V(q_1, \lambda; q_1^0) = 0$, yields the identity

$$V[q_1, \lambda(q_1; q_1^0); q_1^0] \equiv 0 \tag{43}$$

Differentiation of this equation due to the second of Eqs. (41) leads to

$$\frac{\mathrm{d}\lambda}{\mathrm{d}q_1} = 0 \tag{44}$$

Differentiating the identity (43) twice, and using Eq. (44), we get

$$\frac{\mathrm{d}V}{\mathrm{d}\lambda} \cdot \frac{\mathrm{d}^2\lambda}{\mathrm{d}q_1^2} = -\frac{\mathrm{d}^2 V}{\mathrm{d}q_1^2} \tag{45}$$

Note that $\mathrm{d}V/\mathrm{d}\lambda < 0$, as being equal to the minus of displacement. Then inserting the solution $(q_1^D, \tilde{\lambda}_D)$ into the right-hand side of Eq. (45) yields

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}q_1^2} = \frac{\mathrm{d}^2 V}{\mathrm{d}q_1^2}(q_1^D, \tilde{\lambda}_D; q_1^0) < 0 \tag{46}$$

From Eqs. (44) and (46) it is deduced that the point $(q_1^D, \tilde{\lambda}_D)$ is a maximum in the curve λ vs q_1 [see Eq. (42)]. Hence, Eqs. (44) and (46) constitute sufficient conditions for dynamic buckling of undamped one-degree-of-freedom imperfect systems.

We can now discuss the validity of these conditions in the case of bifurcational perfect systems associated with a branching point: 1) unstable symmetric and 2) asymmetric. Indeed, in

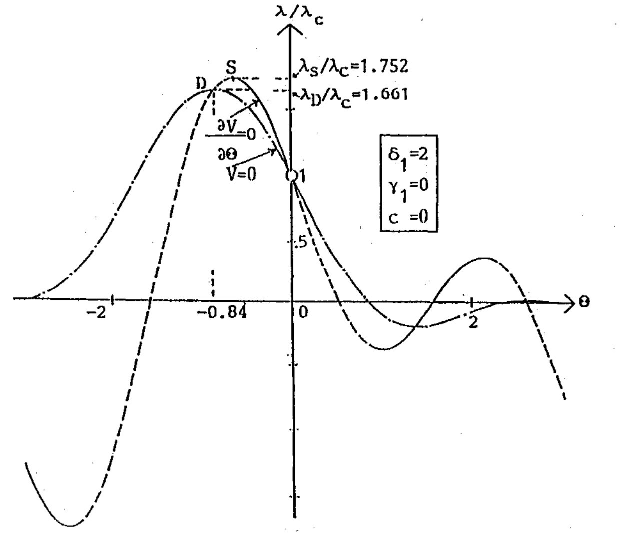

Fig. 3 Static equilibrium path ($\partial V/\partial\Theta = 0$), and dynamic curve λ vs $\Theta(V = 0)$ of a quadratic, one-degree-of-freedom, undamped model ($\delta_1 = 2$, $\gamma_1 = 0$).

case 1 the point ($y^E = 0$, $\lambda = \lambda_c$) satisfies conditions (41) or equivalently conditions (44) and (46); then $q_1^D = 0$ and $\tilde{\lambda}_D = \lambda_c$. However, in case 2 the point ($y^E = 0$, $\lambda = \lambda_c$) is not a maximum in the curve λ vs q_1. The maximum of this curve, if it exists, leading to dynamic buckling, is a saddle point (q_1^D, $\tilde{\lambda}_D$) lying above the branching point (i.e., $\tilde{\lambda}_D > \lambda_c$), which is obtained through the solution of Eqs. (41).

The foregoing findings are supplemented by two numerical examples.

Consider an undamped perfect one-mass model of a simply supported beam[1] under step loading of infinite duration. Equation (40) becomes

$$\ddot{\Theta} + \frac{\partial V}{\partial \Theta} = 0 \tag{47}$$

where

$$V = \frac{1}{2}\sin^2\Theta - \frac{\delta_1}{3}\sin^3\Theta - \frac{Y_1}{4}\sin^4\Theta$$
$$-\frac{\lambda}{\lambda_c}(1 - \cos\Theta) \qquad \text{for} \qquad |\Theta| < \pi/2 \tag{48}$$

and where λ_c is the static bifurcational buckling load, whereas δ_1 and γ_1 are dimensionless nonlinear stiffness coefficients. This system under statically applied load exhibits an asymmetric branching point if $\delta_1 \neq 0$ and $\gamma_1 = 0$, while if $\delta_1 = 0$ and $\gamma_1 \neq 0$ it exhibits a symmetric branching point.

For the case $\delta_1 \neq 0$ and $\gamma_1 = 0$, Eqs. (41), after some elaboration, for $\Theta \neq 0$, are written as follows:

$$\frac{\lambda}{\lambda_c} = \cos^2\frac{\Theta}{2}\left(1 - \frac{2\delta_1}{3}\sin\Theta\right)$$
$$\frac{\lambda}{\lambda_c} = \cos\Theta(1 - \delta_1\sin\Theta) \tag{49}$$

From these equations, after some manipulation, we get

$$\tan\frac{\Theta}{2} = 2\delta_1\left(1 - \frac{4}{3}\cos^2\frac{\Theta}{2}\right) \tag{50}$$

For $\delta_1 > 0$ and $|\Theta| < \pi/2$ (implying $\cos\Theta > 0$), the second of Eqs. (49) for $\lambda/\lambda_c > 0$ yields $\delta_1 \sin\Theta < 1$ or due to Eq. (50)

$$1 > \frac{\tan(\Theta/2)\sin\Theta}{2[1 - (4/3)\cos^2(\Theta/2)]} = \frac{\sin^2(\Theta/2)}{1 - (4/3)\cos^2(\Theta/2)} \tag{51}$$

This inequality implies

$$\cos^2\frac{\Theta}{2} > \frac{3}{4} \tag{52}$$

Equation (50) due to inequality (52) yields $\Theta < 0$, which due to inequality (52) implies $\Theta_D > -\pi/3$.

If $\delta_1 = 2$ and $\gamma_1 = 0$ (quadratic model), we find $\Theta_D = -0.84$ and $\lambda_D/\lambda_c = 1.661$ due to which Eq. (48) yields $V = 6.4 \times 10^{-9} \cong 0$. A graphical representation of this case is shown in Fig. 3.

These analytically obtained results are verified by solving numerically Eq. (47) subject to initial conditions: $\Theta(0) = \dot{\Theta}(0) \simeq 0$ [i.e., $\dot{\Theta}(0) = -10^{-12}$].

From Fig. 4 one can see three phase-plane portraits corresponding to $\lambda/\lambda_c < 1$, $1 < \lambda/\lambda_c < \lambda_D/\lambda_c$, and $\lambda/\lambda_c > \lambda_D/\lambda_c$. It is worth observing that the static bifurcation coincides with the dynamic bifurcation (qualitative change of the response); however, at $\lambda/\lambda_c = 1$ and $\Theta < 0$ the system is statically stable up to $\lambda_s/\lambda_c = 1.752$ but dynamically globally stable only up to $\lambda_D/\lambda_c = 1.661 < \lambda_s/\lambda_c = 1.752$. This important finding reported for the first time in the technical literature is also valid when damping is included. To this end one should add to Eq. (47) $c\dot{\Theta}\cos^2\Theta$. The first integral of the resulting equation for a system initially at rest ($t = 0$) gives by means of Eq. (2) the total energy E

$$E = \tfrac{1}{2}\dot{\Theta}^2 + c\int_0^t \dot{\Theta}^2\cos^2\Theta \, \mathrm{d}t' + V = 0 \tag{53}$$

Since dynamic buckling occurs through a saddle point, Eq. (53) leads to the following condition:

$$V = -c\int_0^t \dot{\Theta}^2\cos^2\Theta \, \mathrm{d}t' \tag{54}$$

Equation (54) for $c \to 0$ implies $V = 0$ that according to Eq. (41) yields the dynamic buckling load $\tilde{\lambda}_D$ for vanishing but nonzero damping. If $c = 0.01$, we find $\Theta_{DD} = -0.837$ and $\lambda_{DD}/\lambda_c = 1.664$. The dynamic critical point lying slightly above the corresponding critical point ($\Theta_D = -0.84$, $\lambda_D/\lambda_c = 1.661$) of the undamped system is, as expected, a saddle point associ-

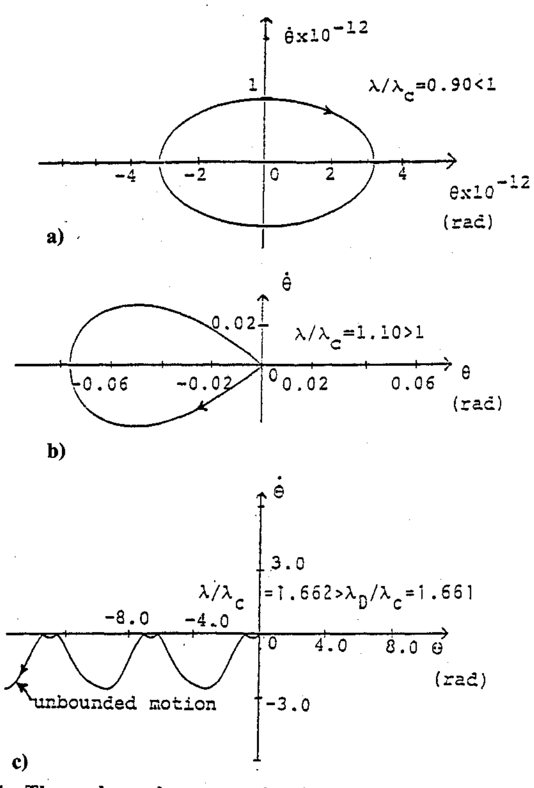

Fig. 4 Three phase-plane portraits a), b), and c) corresponding to three levels of the loading λ/λ_c for $\delta_1 = 2$, $\gamma_1 = 0$ and $c = 0$.

ated with one positive and one negative Jacobian eigenvalue since at such a point $\delta^2 V = d^2V/d\Theta^2 < 0$. From Fig. 5 we can see two phase-plane portraits before and after dynamic buckling, respectively.

The importance of this finding lies in the following fact: although for $\lambda/\lambda_c < \lambda_{DD}/\lambda_c = 1.664$ the motion converges toward the corresponding stable equilibrium points, this does not happen for higher loads. Indeed, for $\lambda/\lambda_c > 1.664$, the system buckles dynamically despite the fact that all equilibrium points on the primary path up to the limit point load $\lambda_s/\lambda_c = 1.752$ are stable, as being associated with eigenvalues having negative real parts. Hence, although the local analysis yields stability, the global one reveals dynamic buckling.

For the case $\delta_1 = 0$ and $\gamma_1 = 0.5$ (cubic model) Eqs. (41), after some elaboration, for $\Theta \neq 0$ or $\Theta \neq \pi$, lead to

$$\frac{\lambda}{\lambda_c} = \cos^2\frac{\Theta}{2}\left(1 - \frac{\gamma_1}{2}\sin^2\Theta\right)$$
$$\frac{\lambda}{\lambda_c} = \cos\Theta(1 - \gamma_1 \sin^2\Theta) \tag{55}$$

From these equations, after some manipulation, we get either $\Theta = 0$ (since $\Theta = 2\pi$ is excluded) or

$$\cos^4\frac{\Theta}{2} - \frac{2}{3}\cos^2\frac{\Theta}{2} + \frac{1}{6\gamma_1} = 0 \tag{56}$$

The only acceptable solution of Eq. (56) is $\Theta = 0$; therefore, $\Theta_D = 0$ and $\lambda_D/\lambda_c = 1$. This solution, which satisfies conditions (44) and (46), is shown graphically in Fig. 6. This result can also be verified by numerical integration of the corresponding equation of motion.

The inclusion of damping does not change the previous solution. Hence, regardless of whether or not damping is included, the static unstable bifurcation (0, 1) for which $\delta^4 V(0, 1) = -3(1 + 2\gamma)\delta\Theta^4 = -6\delta\Theta^6 < 0$ is also a dynamic unstable bifurcation associated with one negative and one positive eigenvalue. Note that the point (0, 1) is stable for $\gamma < -½$ and unstable for $\gamma > -½$.

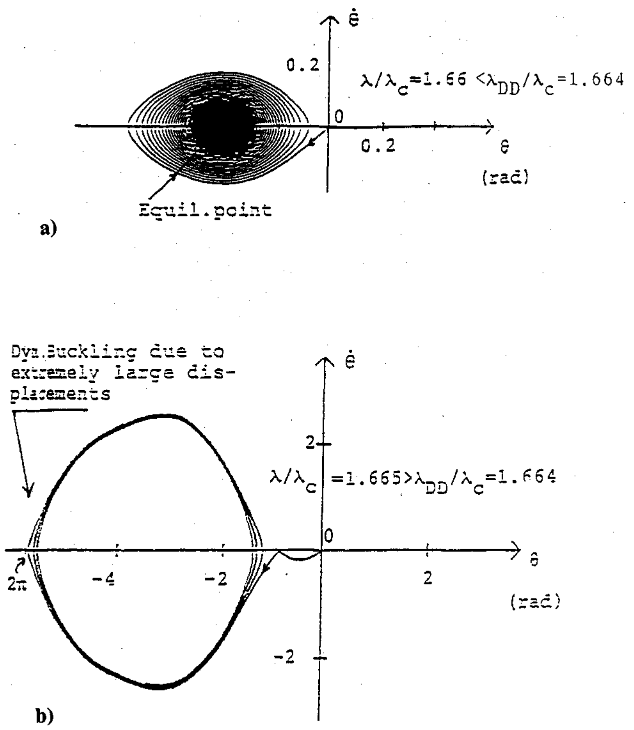

Fig. 5 Two phase-plane portraits for $\delta_1 = 2$, $\gamma_1 = 0$ and $c = 0.01$: a) before dynamic buckling ($\lambda/\lambda_c = 1.66$) and b) after dynamic buckling ($\lambda/\lambda_c = 1.665$).

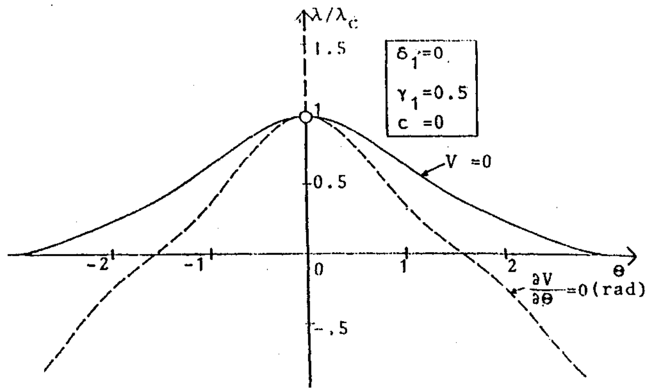

Fig. 6 Static equilibrium path ($\partial V/\partial\Theta = 0$) and dynamic curve λ vs Θ ($V = 0$) of a cubic, one-degree-of-freedom, undamped model ($\delta_1 = 0$, $\gamma_1 = 0.5$, $c = 0$).

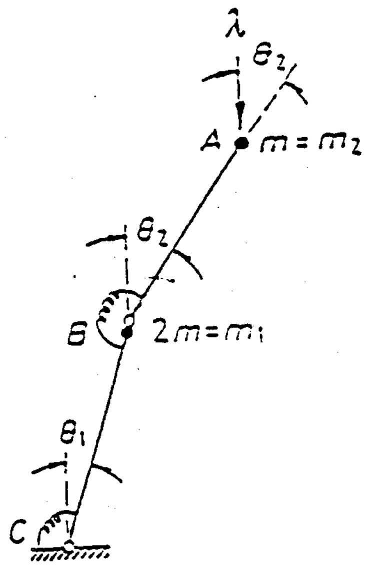

Fig. 7 Ziegler's two-degree-of-freedom model under step loading.

Two-Degree-of-Freedom Systems

Consider a two-degree-of-freedom system that under statically applied load exhibits an unstable branching point on a trivial fundamental path. One can readily write Eq. (10) for $i = 1, 2$ where

$$Q_1^* = \dot{q}^T S_1^* \dot{q} = \dot{q}^T S_1 \dot{q} = \dot{q}_2^2 \frac{\partial \alpha_{12}}{\partial q_2}$$

$$Q_2^* = \dot{q}^T S_2^* \dot{q} = \dot{q}^T S_2 \dot{q} = \dot{q}_1^2 \frac{\partial \alpha_{12}}{\partial q_1}$$

because

$$S_1 = \begin{bmatrix} 0 & \dfrac{1}{2}\dfrac{\partial \alpha_{12}}{\partial q_1} \\ -\dfrac{1}{2}\dfrac{\partial \alpha_{12}}{\partial q_1} & \dfrac{\partial \alpha_{12}}{\partial q_2} \end{bmatrix}, \quad S_2 = \begin{bmatrix} \dfrac{\partial \alpha_{12}}{\partial q_1} & -\dfrac{1}{2}\dfrac{\partial \alpha_{12}}{\partial q_2} \\ \dfrac{1}{2}\dfrac{\partial \alpha_{12}}{\partial q_2} & 0 \end{bmatrix} \tag{57}$$

Setting according to Eq. (11)

$$y_1 = q_1, \quad y_2 = q_2, \quad y_3 = \dot{q}_1, \quad y_4 = \dot{q}_2 \tag{58}$$

Equations (10) by virtue of relations (58) are written as follows:

$$\dot{y}_i = Y_i(y_1, y_2, y_3, y_4; \lambda), \qquad i = 1, 2, 3, 4 \tag{59}$$

where

$$Y_1 = y_3, \qquad Y_2 = y_4$$

$$Y_3 = -b_{11}y_4^2\frac{\partial\alpha_{12}}{\partial q_2} - b_{12}y_3^2\frac{\partial\alpha_{12}}{\partial q_1} - \tilde{c}_{11}y_3 - \tilde{c}_{12}y_4 - b_{11}\frac{\partial V}{\partial y_1} - b_{12}\frac{\partial V}{\partial y_2}$$
$$Y_4 = -b_{21}y_4^2\frac{\partial\alpha_{12}}{\partial q_2} - b_{22}y_3^2\frac{\partial\alpha_{12}}{\partial q_1} - \tilde{c}_{21}y_3 - \tilde{c}_{22}y_4 - b_{21}\frac{\partial V}{\partial y_1} - b_{22}\frac{\partial V}{\partial y_2} \tag{60}$$

where b_{ij} and $\tilde{c}_{ij}$ are elements of the matrices given in Eq. (10). Equations (36) and (37) for $[c_{ij}]\to 0$ yield the following equations in λ, q_1 and q_2:

$$V(q_1, q_2, \lambda) = \frac{\partial V}{\partial q_1}(q_1, q_2, \lambda) = \frac{\partial V}{\partial q_2}(q_1, q_2, \lambda) = 0 \tag{61}$$

The smallest load obtained through Eqs. (61), which renders the second variation $\delta^2 V$ indefinite, is the dynamic buckling load $\tilde{\lambda}_D$ for vanishing but nonzero damping. This load satisfies inequality (34).

Consider, as an illustrative example, the Ziegler two-degree-of-freedom model shown in Fig. 7. If one sets $q_1 = \ell \sin\Theta_1$ and $q_2 = \ell(\sin\Theta_1 + \sin\Theta_2)$, Eqs. (1), in dimensionless form, in terms of Θ_1 and Θ_2 are[2]

$$(1+m)\ddot{\Theta}_1 + \ddot{\Theta}_2\cos(\Theta_1 - \Theta_2) + \dot{\Theta}_2^2\sin(\Theta_1 - \Theta_2) + (c_1 + c_2)\dot{\Theta}_1 - c_2\dot{\Theta}_2 + \frac{\partial V}{\partial\Theta_1} = 0$$
$$\ddot{\Theta}_2 + \ddot{\Theta}_1\cos(\Theta_1 - \Theta_2) - \dot{\Theta}_1^2\sin(\Theta_1 - \Theta_2) + c_2\dot{\Theta}_2 - c_2\dot{\Theta}_1 + \frac{\partial V}{\partial\Theta_2} = 0 \tag{62}$$

where c_1 and c_2 are linear viscous coefficients of the two springs, respectively; $m = m_1/m_2$ is the mass ratio of the two concentrated masses, whereas

$$\frac{\partial V}{\partial\Theta_1} = 2\Theta_1 - \Theta_2 + \delta_1\Theta_1^2 - \delta_2(\Theta_1 - \Theta_2)^2 + \gamma_1\Theta_1^3 + \gamma_2(\Theta_1 - \Theta_2)^3 - \lambda\sin\Theta_1 = 0 \tag{63}$$

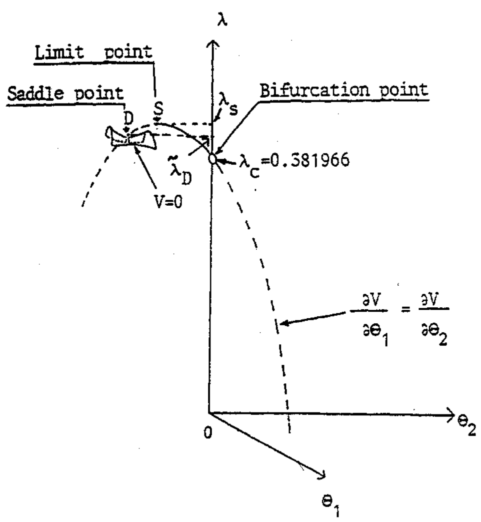

Fig. 8 Equilibrium path ($\partial V/\partial\Theta_1 = \partial V/\partial\Theta_2 = 0$) and $\lambda = \lambda(\Theta_1, \Theta_2)$ ($V = 0$) for a quadratic ($\delta_1 = 1$, $\delta_2 = -10.1$, $\gamma_1 = \gamma_2 = 0$) two degree-of-freedom undamped ($c_1 = c_2 = 0$) system.

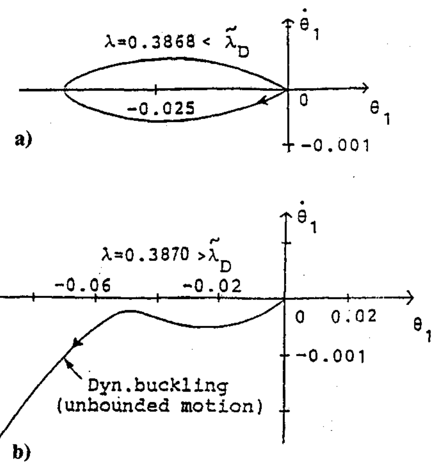

Fig. 9 Two phase-plane portraits a) before and b) after dynamic buckling of a quadratic ($\delta_1 = 1$, $\delta_2 = -10.1$, $\gamma_1 = \gamma_2 = 0$) two-degree-of-freedom system with $c_{ij}\to 0$.

$$\frac{\partial V}{\partial\Theta_2} = -\Theta_1 + \Theta_2 + \delta_2(\Theta_1 - \Theta_2)^2 - \gamma_2(\Theta_1 - \Theta_2)^3 - \lambda\sin\Theta_2 = 0$$

For $|\delta_1| + |\delta_2| \neq 0$ [excluding $\delta_1 + \delta_2(1-\lambda_c)^3 = 0$] and $\gamma_1 = \gamma_2 = 0$ (quadratic model) the system under static load exhibits an asymmetric branching point, whereas for $|\gamma_1| + |\gamma_2| \neq 0$ and $\delta_1 = \delta_2 = 0$ (cubic model) it exhibits a symmetric branching point whose stability depends on γ_1 and γ_2.

The matrices $[\alpha_{ij}]$, $[c_{ij}]$, and $[V_{ij}]$ (with $i, j = 1, 2$) evaluated at the trivial state $y^E = 0$ for $m = 2$ are equal to

$$[\alpha_{ij}] = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad [c_{ij}] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \quad [V_{ij}] = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} \tag{64}$$

The characteristic equation of the Jacobian matrix is given by

$$\rho^4 + a_1\rho^3 + a_2\rho^2 + a_3\rho + a_4 = 0 \tag{65}$$

where

$$a_1 = \tfrac{1}{2}(c_1 + 6c_2), \qquad a_2 = \tfrac{1}{2}(c_1c_2 + 7 - 4\lambda)$$
$$a_3 = \tfrac{1}{2}[(1-\lambda)c_1 + (1-2\lambda)c_2], \qquad a_4 = \tfrac{1}{2}(\lambda^2 - 3\lambda + 1) \tag{66}$$

Since $\delta^2 V$ at the critical state ($\Theta_1 = \Theta_2 = 0$, $\lambda = \lambda_c$) becomes positive semidefinite, due to Eq. (64), we can write

$$\begin{bmatrix} 2-\lambda_c & -1 \\ -1 & 1-\lambda_c \end{bmatrix} \cdot \begin{bmatrix} \delta\Theta_1 \\ \delta\Theta_2 \end{bmatrix} = 0 \tag{67}$$

which for a non-trivial solution yields $\lambda_c = 0.5(3 - \sqrt{5}) = 0.381966$.

Taking also into account that $\delta\Theta_2/\delta\Theta_1 = 2 - \lambda_c = (1-\lambda_c)^{-1}$, one can establish for $\gamma_1 = \gamma_2 = \delta_1 = 0$ and $\delta_2 = -10.1$ the following relationship:

$$\delta^3 V(0, 0, \lambda_c) = [2(\delta_1 - \delta_2) + 6\delta_2(2-\lambda_c) - 6\delta_2(2-\lambda_c)^2 + 2\delta_2(2-\lambda_c)^3]\cdot\delta\Theta_1^3 = -2.77\delta\Theta_1^3 \neq 0 \tag{68}$$

Namely, the critical state ($\Theta_1 = \Theta_2 = 0$, $\lambda = \lambda_c$) is associated with an asymmetric branching point.

If there is no damping ($c_1 = c_2 = 0$), Eqs. (65) and (66) yield $a_1 = a_3 = 0$. At the critical state ($a_4 = 0$), the Jacobian has one double zero eigenvalue and a pair of pure imaginary roots, $\pm\sqrt{a_2} = \pm 1.654i$. For $\lambda_c < \lambda < 0.5(3+\sqrt{5})$, clearly $a_4 < 0$. Then there is one positive eigenvalue equal to $[0.5(-a_2 + \sqrt{a_2^2 - 4a_4})]^{1/2}$; hence the trivial state $\Theta_1 = \Theta_2 = 0$ is unstable.

If damping is included (e.g., $c_1 = c_2 = 0.01$), one can readily show that Eq. (65), evaluated at the critical state, has one zero and one negative eigenvalue as well as a pair of complex conjugate eigenvalues with negative real part. If Eq. (65) is evaluated at the trivial state $\Theta_1 = \Theta_2 = 0$ for $\lambda_c < \lambda < 0.5\cdot(3+\sqrt{5})$, then $a_4 < 0$. Since a_1, a_2, and $a_3 > 0$ and $a_4 < 0$, according to the theory of algebraic equations, Eq. (65) has at least one positive eigenvalue, and hence the trivial state is unstable. Using a local dynamic analysis, one can also prove that all equilibrium points of the prebuckling path up to the limit point load $\lambda_s = 0.3874$ are stable. However, as shown next, this is not true when a nonlinear (global) dynamic analysis is employed. Indeed, the system under step loading of infinite duration may exhibit dynamic buckling for Θ_1 and $\Theta_2 < 0$ at a load $\lambda_c < \lambda < \lambda_s = 0.3874$. The dynamic buckling load $\tilde{\lambda}_D$ for vanishing but nonzero damping, if such a load exists, is obtained by solving the corresponding to Eqs. (61) system of equations with respect to Θ_1, Θ_2, and λ from which we get

$$\tilde{\lambda}_D = 0.3869 < \lambda_s = 0.3874$$
$$\Theta_1^D = -0.049, \qquad \Theta_2^D = -0.0715 \tag{69}$$

A graphical representation of the nonlinear equilibrium path λ vs Θ_1 and Θ_2 in the load-displacement space is shown in Fig. 8.

Solution (69) is verified with the aid of numerical integration of Eqs. (62) for $c_1 = c_2 \to 0$ subject to the initial conditions $\Theta_1(0) = \Theta_2(0) = \dot{\Theta}_1(0) = 0$ and $\dot{\Theta}_2 = -10^{-12}$. In Fig. 9 one can see a phase-plane portrait: 1) before dynamic buckling for $\lambda = 0.3868 < 0.3869$ where the motion converges toward the corresponding stable equilibrium point, and 2) a phase-plane portrait for a load $\lambda = 0.3870$ slightly higher than $\tilde{\lambda}_D$. Hence, dynamic buckling occurs despite the fact that the primary equilibrium path up to $\lambda_s = 0.3874$ is stable according to the local dynamic analysis. The explanation of such a contradiction between global and local dynamic analysis was given earlier in the subsection about the mechanism of dynamic buckling.

Consider now the case $\delta_1 = \delta_2 = 0$ and $|\gamma_1| + |\gamma_2| \neq 0$ (cubic model). The system under static loading exhibits a symmetric branching point at $\lambda_c = 0.381966$ and $\Theta_1 = \Theta_2 = 0$. The equilibrium path is symmetric with respect to the vertical loading axis. Depending on the values of γ_1 and γ_2, there are three characteristic cases of static equilibrium paths. The first two are shown in Fig. 10a, and the third one in Fig. 10b. In the first case ($\gamma_1 = -10$, $\gamma_2 = -10.6$), the critical point is an unstable symmetric bifurcation point. The total potential V vanishes only at this point. Hence, the static bifurcation is also a dynamic bifurcation. The results of the local stability analysis coincide with those of the global analysis. In the second case ($\gamma_1 = \gamma_2 = 1$), the system under static loading exhibits a continuously rising (i.e., stable) equilibrium path, and therefore, under step loading of infinite duration, it experiences a point attractor response.

In the third case ($\gamma_1 = 250$, $\gamma_2 = -10.6$), the system under static loading exhibits a stable symmetric bifurcation point at $\lambda_c = 0.381966$ and thereafter a limit point at $\lambda_s = 0.5527$. The total potential energy V vanishes at the saddle point ($\Theta_1 = 0.069$, $\Theta_2 = 0.225$, $\lambda = 0.5219$). Hence, the system under step load of infinite duration exhibits a dynamic global bifurcation at $\tilde{\lambda}_D = 0.5219$, being the dynamic buckling load for vanishing but nonzero damping. However, according to the local dynamic analysis all prebuckling equilibriums for $\tilde{\lambda}_D < \lambda < \lambda_s$ are stable associated with complex conjugate Jacobian eigenvalues having negative real parts. Figure 11 shows two phase-plane portraits, before and after dynamic buckling.

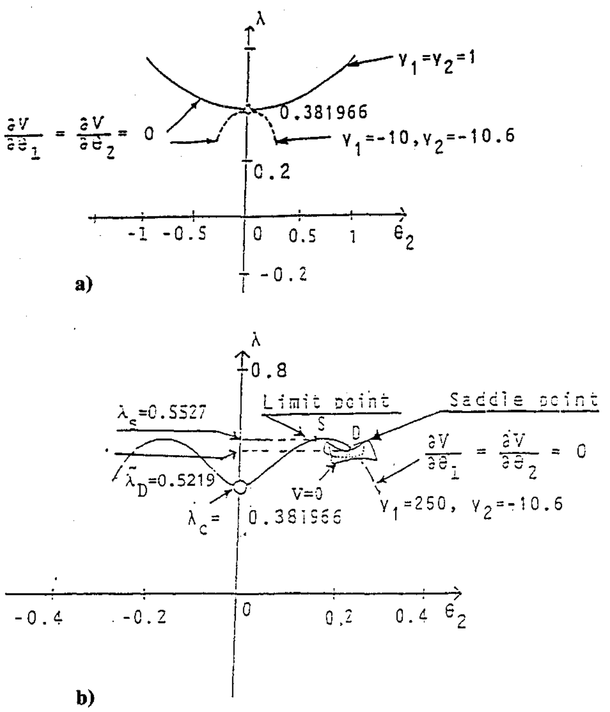

Fig. 10 Three equilibrium paths of a cubic ($\delta_1 = \delta_2 = 0$) two-degree-of-freedom system for: a) $\gamma_1 = -10$, $\gamma_2 = -10.6$, and $\gamma_1 = \gamma_2 = 1$, and b) $\gamma_1 = 250$ and $\gamma_2 = -10.6$.

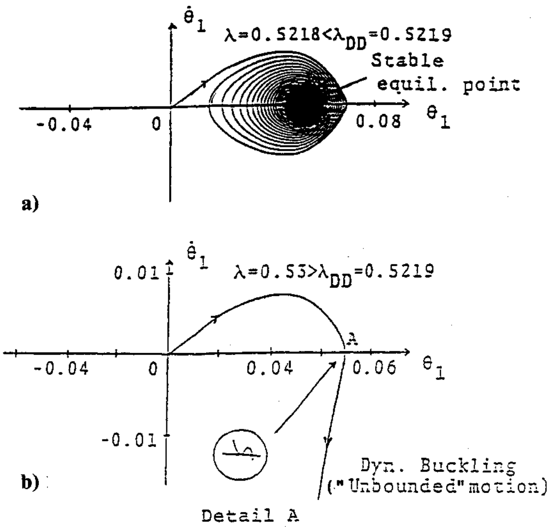

Fig. 11 Phase-plane portraits a) before and b) after dynamic buckling of a cubic ($\gamma_1 = 250$, $\gamma_2 = -10.6$) two-degree-of-freedom system with $c_{ij} \to 0$.

V. Conclusions

The most important findings of this investigation are the following:

1) Treating analytically Lagrange's equations of motion, in their most general form, it is found that the Jacobian matrix is a block matrix composed of four square submatrices properly identified.

2) Dynamic bifurcations and stability of equilibriums in the precritical, critical, and postcritical state are analytically explored by studying the effect of the loading on the nature of the corresponding Jacobian eigenvalues.

3) A relationship between the different types of distinct static bifurcations points (asymmetric, stable and unstable symmetric) with corresponding, possibly existing, dynamic local bifurcations is established.

4) Using the concepts of the basin of attraction of a stable equilibrium point and of the inset and outset manifolds of a saddle, the mechanism of dynamic buckling is theoretically discussed. This refers to general systems that under static loading exhibit either a limit point instability or a bifurcational instability associated with all types of distinct branching points emanating from a trivial fundamental path.

5) Using these concepts, the total potential function and the total energy equation, one can find for the case of vanishing but nonzero damping the saddle point through which occurs an escaped motion leading to dynamic buckling and the exact dynamic buckling load.

6) Perfect bifurcational systems, regardless of the type of their branching point, cannot buckle dynamically at a load less than the static bifurcational load. Hence, the local dynamic analysis gives the same results as those of the global dynamic one.

7) Discrepancies between local and global dynamic analyses may occur in the case of unstable or stable symmetric bifurcational systems that experience a postbifurcational limit point instability. Then stable equilibriums prior to the limit point are found to be globally unstable associated with dynamic buckling.

8) Although static bifurcations can also be dynamic local bifurcations, the contrary is not true. Indeed, in the case of limit point instability, dynamic buckling occurs before static buckling.

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